CENTRALIZERS OF CERTAIN QUADRATIC ELEMENTS IN POISSON–LIE ALGEBRAS AND ARGUMENT SHIFT METHOD

L. G. RYBNIKOV

1. Introduction.

Let $\mathfrak{g}$ be a semisimple complex Lie algebra. The universal enveloping algebra $U(\mathfrak{g})$ bears a natural filtration by the degree with respect to the generators. The associated graded algebra $\operatorname{gr} U(\mathfrak{g})$ is naturally isomorphic to the symmetric algebra $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$ by the Poincaré–Birkhoff–Witt theorem. The commutator operation on $U(\mathfrak{g})$ defines a Poisson bracket on $S(\mathfrak{g})$, which we call the *Poisson–Lie bracket*.

The *argument shift method* gives a way to construct Poisson-commutative subalgebras in $S(\mathfrak{g})$. The method is as follows. Let $ZS(\mathfrak{g}) = S(\mathfrak{g})^{\mathfrak{g}}$ be the center of $S(\mathfrak{g})$ with respect to the Poisson bracket, and let $\mu \in \mathfrak{g}^*$ be a regular semisimple element. Then the algebra $A_\mu \subset S(\mathfrak{g})$ generated by the elements $\partial_\mu^n \Phi$, where $\Phi \in ZS(\mathfrak{g})$, (or, equivalently, generated by central elements of $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$ shifted by $t\mu$ for all $t \in \mathbb{C}$) is Poisson-commutative and has maximal possible transcendence degree equal to $\frac{1}{2}(\dim \mathfrak{g} + \operatorname{rk} \mathfrak{g})$ (see [3]). Moreover, the subalgebras A_μ are maximal Poisson-commutative subalgebras in $S(\mathfrak{g})$ (see [7]). In [9], the subalgebras $A_\mu \subset S(\mathfrak{g})$ are named the *Mischenko–Fomenko subalgebras*.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}$. We denote by Δ and Δ_+ the root system of $\mathfrak{g}$ and the set of positive roots, respectively. Let $\alpha_1, \dots, \alpha_l$ be the simple roots. Fix a non-degenerate invariant scalar product $(\cdot,\cdot)$ on $\mathfrak{g}$ and choose from each root space $\mathfrak{g}_\alpha$, $\alpha \in \Delta$, a nonzero element e_α such that $(e_\alpha, e_{-\alpha}) = 1$. Set $h_\alpha := [e_\alpha, e_{-\alpha}]$, then for any $h \in \mathfrak{h}$ we have $(h_\alpha, h) = \langle \alpha, h \rangle$. The elements e_α $(\alpha \in \Delta)$ together with $h_{\alpha_1}, \dots, h_{\alpha_l} \in \mathfrak{h}$ form a basis of $\mathfrak{g}$.

We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the scalar product $(\cdot,\cdot)$ and assume that μ is a regular semisimple element of the fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g} = \mathfrak{g}^*$. The linear and quadratic part of the Mischenko–Fomenko subalgebras can be described as follows [2]:

$$A_\mu \cap \mathfrak{g} = \mathfrak{h},$$
$$A_\mu \cap S^2(\mathfrak{g}) = S^2(\mathfrak{h}) \oplus Q_\mu, \text{ where } Q_\mu = \{\sum_{\alpha \in \Delta_+} \frac{\langle \alpha, h \rangle}{\langle \alpha, \mu \rangle} e_\alpha e_{-\alpha} | h \in \mathfrak{h}\}.$$

The main result of the present paper is the following

Theorem 1. *For generic $\mu \in \mathfrak{h}$ (i.e. for μ in the complement to a certain countable union of Zariski-closed subsets in $\mathfrak{h}$), the algebra A_μ is the Poisson centralizer of the subspace Q_μ in $S(\mathfrak{g})$.*

The work was partially supported by RFBR grant 04-01-00702, RFBR grant 05 01 00988-a and RFBR grant 05-01-02805-CNRSL-a.

In [1, 4, 6] the Mischenko–Fomenko subalgebras were lifted (quantized) to the universal enveloping algebra, i.e. the family of commutative subalgebras $\mathcal{A}_\mu \subset U(\mathfrak{g})$ such that $\operatorname{gr}\mathcal{A}_\mu = A_\mu$ was constructed for any classical Lie algebra $\mathfrak{g}$ (i.e. $\mathfrak{sl}_r$, $\mathfrak{so}_r$, $\mathfrak{sp}_{2r}$). In [5] we do this (by different methods) for any semisimple $\mathfrak{g}$.

We deduce the following assertion from Theorem 1.

Theorem 2. *For generic $\mu \in \mathfrak{h}$ there exist no more than one commutative subalgebra $\mathcal{A}_\mu \subset U(\mathfrak{g})$ satisfying $\operatorname{gr}\mathcal{A}_\mu = A_\mu$.*

This means that there is a *unique* quantization of Mischenko–Fomenko subalgebras. In particular, the methods of [1, 4, 6] and [5] give the same for classical Lie algebras. In the case $\mathfrak{g} = \mathfrak{gl}_n$ the assertion of Theorem 2 was proved by A. Tarasov [8] for any regular $\mu \in \mathfrak{h}$.

I thank E. B. Vinberg for attention to my work and useful discussions.

2. Proof of Theorem 1

Note that the set $E_n \subset \mathfrak{h}$ of such $\mu \in \mathfrak{h}$ that the Poisson centralizer of the space Q_μ in $S^n(\mathfrak{g})$ has the dimension greater than $\dim A_\mu \cap S^n(\mathfrak{g})$ is Zariski-closed in $\mathfrak{h}$ for any n. Therefore it suffices to prove that $E_n \neq \mathfrak{h}$ for any n. Thus, it suffices to prove the existence of $\mu \in \mathfrak{h}$ satisfying the conditions of the Theorem.

Lemma 1. *There exist $\mu, h \in \mathfrak{h}$ such that numbers $\frac{\langle\alpha,h\rangle}{\langle\alpha,\mu\rangle}$ $(\alpha \in \Delta_+)$ are linearly independent over $\mathbb{Q}$.*

Proof. Choose μ such that the values $\alpha_i(\mu)$ are algebraically independent over $\mathbb{Q}$ for simple roots α_i. Since there are no proportional positive roots, the numbers $\frac{1}{\langle\alpha,\mu\rangle}$, $\alpha \in \Delta_+$, are linearly independent over $\mathbb{Q}$. Choose h such that the values $\langle\alpha, h\rangle$ are nonzero rational numbers. Then the numbers $\frac{\langle\alpha,h\rangle}{\langle\alpha,\mu\rangle}$, $\alpha \in \Delta_+$, are linearly independent over $\mathbb{Q}$. $\square$

Choose $\gamma \in \mathfrak{g}^*$ such that $\gamma(h_{\alpha_i}) = 1$ for any simple root α_i and $\gamma(e_\alpha) = 0$ for $\alpha \in \Delta$. We define a new Poisson bracket $\{\cdot,\cdot\}_\gamma$ on $S(\mathfrak{g})$ by setting $\{x, y\}_\gamma = \gamma([x, y])$ for $x, y \in \mathfrak{g}$. This bracket is compatible with the Poisson–Lie bracket, i.e. the linear combination $t\{\cdot,\cdot\} + (1-t)\{\cdot,\cdot\}_\gamma$ is a Poisson bracket on $S(\mathfrak{g})$ (i.e. satisfies the Jacobi identity) for any $t \in \mathbb{C}$. Moreover, for $t \neq 0$, the corresponding Poisson algebras are isomorphic. Namely, denote by $S(\mathfrak{g})_t$ the algebra $S(\mathfrak{g})$ equipped with the Poisson bracket $t\{\cdot,\cdot\} + (1-t)\{\cdot,\cdot\}_\gamma$; then for $t \neq 0$ the Poisson algebra isomorphism $\psi_t : S(\mathfrak{g})_1 \to S(\mathfrak{g})_t$ is defined on the generators $x \in \mathfrak{g}$ as follows: $\psi_t(x) = t^{-1}x + t^{-2}(1-t)\gamma(x)$. Clearly, we have $\psi_t(Q_\mu) = Q_\mu$.

Lemma 2. *The transcendence degree of the Poisson centralizer of the subspace Q_μ in $S(\mathfrak{g})_0$ is not greater than $\frac{1}{2}(\dim\mathfrak{g} + \operatorname{rk}\mathfrak{g})$ for some $\mu \in \mathfrak{h}$.*

Proof. Choose μ and h as in Lemma 1 and set $q = \sum_{\alpha\in\Delta_+} \frac{\langle\alpha,h\rangle}{\langle\alpha,\mu\rangle} e_\alpha e_{-\alpha} \in Q_\mu$. For any $f \in S(\mathfrak{g})$, we have $\{q, f\}_\gamma = \sum_{\alpha\in\Delta_+} \gamma(h_\alpha) \frac{\langle\alpha,h\rangle}{\langle\alpha,\mu\rangle} (e_{-\alpha}\frac{\partial f}{\partial e_{-\alpha}} - e_\alpha \frac{\partial f}{\partial e_\alpha})$. In particular,

$$(1) \quad \{q, \prod_{i=1}^{l} h_{\alpha_i}^{m_i} \prod_{\alpha\in\Delta_+} e_\alpha^{n_\alpha} e_{-\alpha}^{n_{-\alpha}}\}_\gamma =$$

$$= \sum_{\alpha\in\Delta_+} \gamma(h_\alpha) \frac{\langle\alpha,h\rangle}{\langle\alpha,\mu\rangle} (n_{-\alpha} - n_\alpha) \prod_{i=1}^{l} h_{\alpha_i}^{m_i} \prod_{\alpha\in\Delta_+} e_\alpha^{n_\alpha} e_{-\alpha}^{n_{-\alpha}}.$$

For any $\alpha = \sum_{i=1}^{l} k_i\alpha_i \in \Delta_+$, we have $\gamma(h_\alpha) = \sum_{i=1}^{l} k_i \in \mathbb{Q}\backslash\{0\}$. Since the numbers $\frac{\langle\alpha,h\rangle}{\langle\alpha,\mu\rangle}$ are linearly independent over $\mathbb{Q}$, the right hand part of (1) is zero iff $n_\alpha - n_{-\alpha} = 0$ for any $\alpha \in \Delta_+$. This means that the Poisson centralizer of q in $S(\mathfrak{g})_0$ is linearly generated by monomials having equal degrees in e_α and $e_{-\alpha}$ for any $\alpha \in \Delta_+$, i.e. the Poisson centralizer of q in $S(\mathfrak{g})_0$ is generated (as a commutative algebra) by the elements h_{α_i} $(i = 1, \dots, l)$ and $e_\alpha e_{-\alpha}$ $(\alpha \in \Delta_+)$. Therefore, the transcendence degree of the Poisson centralizer of q in $S(\mathfrak{g})_0$ is equal to $\frac{1}{2}(\dim\mathfrak{g} + \mathrm{rk}\,\mathfrak{g})$. □

By Lemma 2, the transcendence degree of the Poisson centralizer of the subspace Q_μ in $S(\mathfrak{g})_t$ is not greater than $\frac{1}{2}(\dim\mathfrak{g} + \mathrm{rk}\,\mathfrak{g})$ for generic t. Since the Poisson algebras $S(\mathfrak{g})_t$ are isomorphic to each other for $t \neq 0$, this lower bound of the transcendence degree holds for any $t \in \mathbb{C}$. Let $Z \subset S(\mathfrak{g})$ be the Poisson centralizer of Q_μ in $S(\mathfrak{g})_1$. Since $\mathrm{tr\ deg}(Z) \le \mathrm{tr\ deg}(A_\mu)$ and $A_\mu \subset Z$, we see that each element of Z is algebraic over A_μ. By Tarasov's results [7], the subalgebra A_μ is algebraically closed in $S(\mathfrak{g})_1$, hence, $Z = A_\mu$. Theorem 1 is proved.

3. Proof of Theorem 2

By [9], the subspace $A_\mu^{(2)} = \mathbb{C} + \mathfrak{h} + S^2(\mathfrak{h}) + Q_\mu \subset S(\mathfrak{g})^{(2)}$ can be uniquely lifted to a commutative subspace $\mathcal{A}_\mu^{(2)} \subset U(\mathfrak{g})^{(2)}$ (this subspace is the image of $A_\mu^{(2)}$ under the symmetrization map). By Theorem 1, any lifting $\mathcal{A}_\mu \subset U(\mathfrak{g})$ of A_μ is the centralizer of the subspace $\mathcal{A}_\mu^{(2)}$ in $U(\mathfrak{g})$ in $U(\mathfrak{g})$ for generic μ. Theorem 2 is proved.

References

[1] Chervov, A. and Talalaev, D. *Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence*, preprint hep-th/0604128

[2] Fomenko, A. T. *Symplectic structures and integrable systems in symmetric spaces. (Russian)* Mat. Sb. (N.S.) 115(157) (1981), no. 2, 263–280, 320. .

[3] Mischenko, A. S.; Fomenko, A. T. *Integrability of Euler's equations on semisimple Lie algebras. (Russian)* Trudy Sem. Vektor. Tenzor. Anal. No. 19 (1979), 3–94.

[4] Nazarov M., Olshanski G., *Bethe Subalgebras in Twisted Yangians*, Comm. Math. Phys., V.178, P.433–506 (1996). q-alg/9507003.

[5] Rybnikov, L. G., *Argument shift method and Gaudin model*, Func. Anal. Appl., **40** (2006), No 3. math.RT/0606380.

[6] Tarasov, A. A. *On some commutative subalgebras in the universal enveloping algebra of the Lie algebra $gl(n, C)$. (Russian)* Mat. Sb. 191 (2000), no. 9, 115–122; translation in Sb. Math. 191 (2000), no. 9-10, 1375–1382.

[7] Tarasov, A. A. *The maximality of some commutative subalgebras in Poisson algebras of semisimple Lie algebras. (Russian)* Uspekhi Mat. Nauk 57 (2002), no. 5(347), 165–166; translation in Russian Math. Surveys 57 (2002), no. 5, 1013–1014.

[8] Tarasov, A. A. *On the uniqueness of the lifting of maximal commutative subalgebras of the Poisson-Lie algebra to the enveloping algebra. (Russian)* Mat. Sb. 194 (2003), no. 7, 155–160; translation in Sb. Math. 194 (2003), no. 7-8, 1105–1111.

[9] Vinberg, E. B., *Some commutative subalgebras of a universal enveloping algebra. (Russian)*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 1, 3–25, 221; translation in Math. USSR-Izv. **36** (1991), no. 1, 1–22.

Poncelet laboratory (Independent University of Moscow and CNRS) and Moscow State University, department of Mechanics and Mathematics

E-mail address: leo.rybnikov@gmail.com